Annals of Studies in Science and Humanities
Volume 1 Number 2 (2015) : 60–74

http://journal.carsu.edu.ph/
Online ISSN 02408-3631

On the Distribution of the Sums, Products and Quotient of Lomax Distributed Random Variables Based on FGM Copula

Milburn Macalos[1,*] *and Jayrold P. Arcede*[2]

[1]Department of Mathematics, Caraga State University, Butuan City, Philippines
[2]Department of Mathematics, Saint Joseph Institute of Technology, Butuan City, Philippines

Received: September 15, 2014   Accepted: September 23, 2015

ABSTRACT

In this article, a Lomax distribution based on Farlie-Gumbel-Morgenstern copula is introduced. Derivations of exact distribution $R = X + Y$, $V = XY$ and $Z = X/(X + Y)$ are obtained in closed form. Corresponding moment properties of these distributions are also derived. The expressions turn out to involve known special functions.

Keywords: Gauss Hypergeometric function, Lomax distribution, products of random variables, quotient of random variables, sum of random variables.

1 Introduction

Copula from the latin word *copulare* means to connect or to join (Sklar, 1959). Essentially, copulas' are functions that join or "couple" multivariate distributions to their one-dimensional marginal distribution functions (Nelsen, 1999). Its sole purpose is to describe the interdependence of several random variables (Schmidt, 2006). A copula is a joint distribution function of the uniform marginals (Nelsen, 2003). When marginals are uniform, they are independent. This implies a flat probability density function and any deviation will indicate dependency (Hutchinson and Lai, 2009).

* Corresponding Author
Email:milburn.macalos@yahoo.com

To date, there has been growing interest in copula owing to its usefulness and popularity though not exempt of criticism (Mikosh, 2006). A listing of copula can be found in Hutchinson and Lai (2009), Joe (1997, ch. 5), and Nelsen (2006: 116-119).

In this study, a Farlie-Gumbel-Morgenstern (FGM) copula is considered. Let $F_X(x)$ and $F_Y(y)$ be the distribution functions of the random variables X and Y, respectively, and θ, $-1 < \theta < 1$, then the probability density function of the bivariate FGM is given by

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) \left[1 + \theta \left(2F_X(x) - 1\right)\left(2F_Y(y) - 1\right)\right] \tag{1}$$

where $f_X(x)$ and $f_Y(y)$ are the pdf's of random variable X and Y respectively. The parameter θ is known as the dependence parameter of X and Y

The FGM copula was first proposed by Morgenstern (1956). According to Trivedi and Zimmer (2007) it is a perturbation of the product copula. It is also attractive due to its simplicity and tractability. Observe that when θ in (1) equals zero, FGM copula collapses to independence. However, FGM copula is restrictive in the sense that dependency of two marginals should be modest in magnitude (Mukherjee et al., 2012). An extensive applications on FGM with varying marginals can be found in Hutchinson and Lai (2009, ch. 2).

Nadarajah (2005) similar to their other works (Nadarajah & Espejo, 2006; Nadarajah & Kotz, 2007) concern on obtaining exact distributions on the sum, product and quotient of some known bivariate distributions. In this note, a bivariate Lomax distribution based on FGM copula is introduced. As to our knowledge, there is still no research done with this marginal.

The paper is organized as follows. Section 2 is devoted on derivations of explicit expressions for the pdfs of $R = X + Y$, $V = XY$ and $Z = X/(X + Y)$, resp. while section 3 is devoted in derivation of raw moments of all pdfs obtained in section 2.

The calculations of this paper involve several special functions. This includes the incomplete beta function

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1} dt,$$

and, the Gauss Hypergeometric function

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!},$$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial. The following results which can be found in Nadarajah and Espejo (2006) are needed in the subsequent discussions.

LEMMA 1 (Nadarajah and Espejo (2006)). *For any $\rho > \alpha > 0$,*

$$\int_0^{\infty} \frac{s^{\alpha-1}}{(s+z)^{\rho}} ds = z^{\alpha-\rho} B(\alpha, \rho - \alpha), \quad z \in \mathbb{R}. \tag{2}$$

where

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$$

for $a > 0$ and $b > 0$ is the beta function.

LEMMA 2. *For $0 < \alpha < \rho + \lambda$,*

$$\begin{aligned}\int_0^\infty x^{\alpha-1}(x+y)^{-\rho}(x+z)^{-\lambda}dx \\ = z^{-\lambda}y^{\alpha-\rho}B(\alpha,\rho+\lambda-\alpha)_2F_1\left(\alpha,\lambda;\rho+\lambda;1-\frac{y}{z}\right).\end{aligned} \tag{3}$$

LEMMA 3. *For $p > 0$ and $q > 0$,*

$$\begin{aligned}\int_a^b (x-a)^{p-1}(b-x)^{q-1}dx \\ = z(b-a)^{p+q-1}(ac+d)^r B(p,q)_2F_1\left(p,-r;p+q;\frac{c(a-b)}{ac+d}\right).\end{aligned} \tag{4}$$

2 Pdfs

Let X and Y be two independent Lomax distributed random variables with probability density functions (pdf) given by

$$f_X(x;\alpha,\theta) = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}; \quad x > 0,\, \alpha > 0,\, \theta > 0 \tag{5}$$

and

$$f_Y(y;\alpha,\theta) = \frac{\alpha\theta^\alpha}{(y+\theta)^{\alpha+1}}; \quad y > 0,\, \alpha > 0,\, \theta > 0, \tag{6}$$

respectively.

The cumulative distribution functions (cdf) of X and Y are known to be

$$F_X(x;\alpha,\theta) = 1 - \left(\frac{\theta}{x+\theta}\right)^\alpha; \quad x > 0,\, \alpha > 0,\, \theta > 0 \tag{7}$$

and

$$F_Y(y;\alpha,\theta) = 1 - \left(\frac{\theta}{y+\theta}\right)^\alpha; \quad y > 0,\, \alpha > 0,\, \theta > 0, \tag{8}$$

respectively.

The following result is the joint pdf derived from FGM copula using Lomax distribution as marginals. It will be used often in this paper as our random variables follows this joint density.

Theorem 2.1. *Let X and Y be random variables that follows Lomax distribution with pdfs in (5) and (6) and cdfs in (7) and (8), respectively. Then the joint density function is given by*

$$f_{X,Y}(x,y;\alpha,\theta;\rho)=\frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}}\frac{\alpha\theta^{\alpha}}{(y+\theta)^{\alpha+1}}\left[1+\rho\left(2\Big(\frac{\theta}{x+\theta}\Big)^{\alpha}-1\right)\left(2\Big(\frac{\theta}{y+\theta}\Big)^{\alpha}-1\right)\right] \quad (9)$$

where x, y, α, θ are all positive and $|\rho|\leq 1$.

Proof. Plugging-in equations (5)–(6) in the FGM copula, we have

$$f_{X,Y}(x,y;\alpha,\theta;\rho)=\frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}}\frac{\alpha\theta^{\alpha}}{(y+\theta)^{\alpha+1}}\left[1+\rho\left(2\Big(\frac{\theta}{x+\theta}\Big)^{\alpha}-1\right)\left(2\Big(\frac{\theta}{y+\theta}\Big)^{\alpha}-1\right)\right].$$

It can be shown that

$$f_{X,Y}(x,y;\alpha,\theta;\rho)\geq 0.$$

We are left to show that

$$\int_0^{\infty}\int_0^{\infty} f_{X,\,Y}\,(x,y;\alpha,\theta;\rho)\,dxdy=1.$$

Now, consider the following

$$\int_0^{\infty}\frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}}\left[2\left(\frac{\theta}{x+\theta}\right)^{\alpha}-1\right]dx=\int_0^{\infty}\frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}}\left[1-2\left(1-\left(\frac{\theta}{x+\theta}\right)^{\alpha}\right)\right]dx$$
$$=1-\int_0^{\infty}\frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}}2\left(1-\left(\frac{\theta}{x+\theta}\right)^{\alpha}\right)dx.$$

Let $u=1-\left(\frac{\theta}{x+\theta}\right)^{\alpha}$. Then $du=\frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}}dx$. If $x=0$, then $u=0$. As $x\to\infty$, $u\to 1$. Hence,

$$1-\int_0^{\infty}\frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}}2\left(1-\left(\frac{\theta}{x+\theta}\right)^{\alpha}\right)dx=1-\int_0^{1}2udu=0.$$

Thus,

$$\int_0^{\infty}\int_0^{\infty}\frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}}\rho\left[2\left(\frac{\theta}{x+\theta}\right)^{\alpha}-1\right]\left[2\left(\frac{\theta}{y+\theta}\right)^{\alpha}-1\right]dxdy=0.$$

Also

$$\int_0^{\infty}\int_0^{\infty}\frac{(\alpha\theta^{\alpha})^2}{[(x+\theta)\,(y+\theta)]^{\alpha+1}}dxdy=1.$$

Consequently, we have

$$\int_0^{\infty}\int_0^{\infty} f_{X,\,Y}\,(x,y;\alpha,\theta;\rho)\,dxdy=1.$$

□

The following figure illustrates the pdf in (9) for specific values: $\alpha = .12$, $\theta = 2$, $\rho = 0.5$.

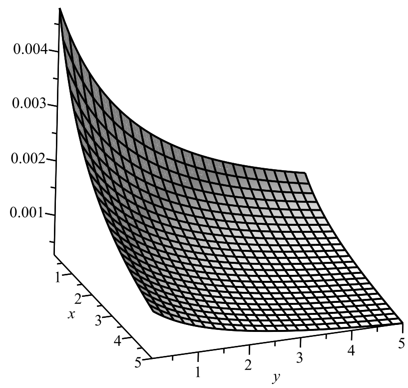

Fig. 1: Graph of the pdf in (9)

Theorems (2.2)–(2.5) derive the pdfs of $R = X + Y$, $V = XY$ and $W = X/(X + Y)$ when X and Y are distributed according to (9). In the subsequent, we assume that α, θ are positive real numbers and $\rho \in [-1, 1]$.

Theorem 2.2. *If X and Y are jointly distributed according to (9), then the density function of $V = XY$ is*

$$\begin{aligned} f_V(v;\alpha,\theta;\rho) = (\alpha\theta^\alpha)^2 \Bigg[& \frac{(1+\rho)}{v^{\alpha+1}} B(\alpha+1,\alpha+1)\,{}_2F_1\left(\alpha+1,\alpha+1;2\alpha+2;1-\frac{\theta^2}{v}\right) \\ & + \frac{4\rho\theta^{2\alpha}}{v^{2\alpha+1}} B(2\alpha+1,2\alpha+1)\,{}_2F_1\left(2\alpha+1,2\alpha+1;4\alpha+2;1-\frac{\theta^2}{v}\right) \\ & - \frac{2\rho}{v^{\alpha+1}} B(\alpha+1,2\alpha+1)\,{}_2F_1\left(\alpha+1,\alpha+1;3\alpha+2;1-\frac{\theta^2}{v}\right) \\ & - \frac{2\rho\theta^{2\alpha}}{v^{2\alpha+1}} B(2\alpha+1,\alpha+1)\,{}_2F_1\left(2\alpha+1,2\alpha+1;3\alpha+2;1-\frac{\theta^2}{v}\right) \Bigg] \end{aligned} \tag{10}$$

for $0 < v < \infty$.

Proof. From (9), the joint pdf of $(X, Y) = \left(X, \frac{V}{X}\right)$ can be expressed as

$$f_{X,V}\left(x, \frac{v}{x}; \alpha, \theta; \rho\right) = (\alpha\theta^{\alpha})^2 \Bigg\{ \frac{1+\rho}{\left[(x+\theta)(\frac{v}{x}+\theta)\right]^{\alpha+1}} + \frac{4\rho\theta^{2\alpha}}{\left[(x+\theta)(\frac{v}{x}+\theta)\right]^{2\alpha+1}} - \frac{2\rho\theta^{\alpha}}{(x+\theta)^{2\alpha+1}\left(\frac{v}{x}+\theta\right)^{\alpha+1}} - \frac{2\rho\theta^{\alpha}}{(x+\theta)^{\alpha+1}\left(\frac{v}{x}+\theta\right)^{2\alpha+1}} \Bigg\}.$$

By Rohatgi's well-known result (1976, p. 141), the pdf of $V = XY$ becomes

$$f_V(v; \alpha, \theta; \rho) = (\alpha\theta^{\alpha})^2 \left[(1+\rho) A(1,1) + 4\rho\theta^{2\alpha} A(2,2) - 2\rho\theta^{\alpha} A(2,1) - 2\rho\theta^{\alpha} A(1,2)\right] \quad (11)$$

where

$$A(h,k) = \int_0^{\infty} x^{k\alpha} (x+\theta)^{-(h\alpha+1)} (v + \theta \cdot x)^{-(k\alpha+1)} \, dx, \quad \text{for } h, k \in \{1, 2\}.$$

Using Lemma (2) we obtain

$$A(h,k) = \theta^{k\alpha - h\alpha} v^{-(k\alpha+1)} B(k\alpha+1, h\alpha+1) \; {}_2F_1\left(k\alpha+1, k\alpha+1; (h+k)\alpha+2; 1 - \frac{\theta^2}{v}\right). \quad (12)$$

Applying (12) to the equation (11) will result to (10). □

Figure below illustrate the shape of the pdf in (10) for $\theta = 2, 4$. Each plot contains three curves corresponding to selected values of α. The effect of the parameters is evident.

Theorem 2.3. *If X and Y are jointly distributed according to (9), then the distribution of $W = \frac{X}{Y}$ is*

$$\begin{aligned} f_W(w; \alpha, \theta; \rho) = (\alpha)^2 \Big[&(1+\rho) B(2, 2\alpha) \; {}_2F_1\left(2, \alpha+1; 2\alpha+2; 1 - w^{-1}\right) \\ &+ 4\rho B(2, 4\alpha) \; {}_2F_1\left(2, 2\alpha+1; 4\alpha+2; 1 - w^{-1}\right) \\ &- 2\rho B(2, 3\alpha) \; {}_2F_1\left(2, \alpha+1; 3\alpha+2; 1 - w^{-1}\right) \\ &- 2\rho B(2, 3\alpha) \; {}_2F_1\left(2, 2\alpha+1; 3\alpha+2; 1 - w^{-1}\right) \Big]. \end{aligned} \quad (13)$$

for $0 < w < \infty$.

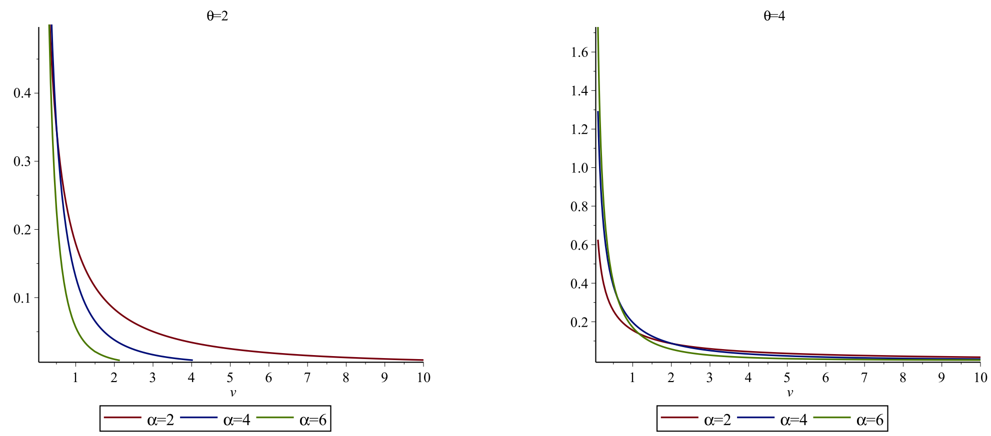

Fig. 2: Graph of the pdf in (10) with selected values of θ and α.

Proof. From (9), the joint pdf of $(X,Y)=\left(X,\frac{X}{W}\right)$ can be expressed as

$$f_{X,W}\left(x,\frac{x}{w};\alpha,\theta;\rho\right)=(\alpha\theta^{\alpha})^2\left\{\frac{1+\rho}{\left[(x+\theta)(\frac{x}{w}+\theta)\right]^{\alpha+1}}+\frac{4\rho\theta^{2\alpha}}{\left[(x+\theta)(\frac{x}{w}+\theta)\right]^{2\alpha+1}}\right.$$
$$\left.-\frac{2\rho\theta^{\alpha}}{(x+\theta)^{2\alpha+1}\left(\frac{x}{w}+\theta\right)^{\alpha+1}}-\frac{2\rho\theta^{\alpha}}{(x+\theta)^{\alpha+1}\left(\frac{x}{w}+\theta\right)^{2\alpha+1}}\right\}.$$

By Rohatgi's result, the pdf of $W=\frac{X}{Y}$ can be expressed as

$$f_W\left(w;\alpha,\theta;\rho\right)=(\alpha\theta^{\alpha})^2\left[(1+\rho)\,C(1,1)+4\rho\theta^{2\alpha}C(2,2)-2\rho\theta^{\alpha}C(2,1)-2\rho\theta^{\alpha}C(1,2)\right] \tag{14}$$

where

$$C(h,k)=\int_0^{\infty}w^{k\alpha+1}x\,(x+\theta)^{-(h\alpha+1)}\,(x+\theta\cdot w)^{-(k\alpha+1)}\,dx. \tag{15}$$

for $h,k\in\{1,2\}$.

Using Lemma (2) one can get

$$C(h,k)=\theta^{-(h+k)\alpha}B\left(2,(h+k)\alpha\right)\ {}_2F_1\left(2,k\alpha+1;(h+k)\alpha+2;1-w^{-1}\right). \tag{16}$$

By (16), the following terms in (14) are obvious.

(1) $(1+\rho)\,C(1,1)=(1+\rho)\theta^{-2\alpha}B\left(2,2\alpha\right)\ {}_2F_1\left(2,\alpha+1;2\alpha+2;1-w^{-1}\right)$;

(2) $4\rho\theta^{2\alpha}C(2,2)=4\rho\theta^{-2\alpha}B\left(2,4\alpha\right)\ {}_2F_1\left(2,2\alpha+1;4\alpha+2;1-w^{-1}\right)$;

(3) $-2\rho\theta^{\alpha}C(2,1) = -2\rho\theta^{-2\alpha}B\left(2,3\alpha\right)\ {}_2F_1\left(2,\alpha+1;3\alpha+2;1-w^{-1}\right);$

(4) $-2\rho\theta^{\alpha}C(1,2) = -2\rho\theta^{-2\alpha}B\left(2,3\alpha\right)\ {}_2F_1\left(2,2\alpha+1;3\alpha+2;1-w^{-1}\right);$

The result follows by using items (1)–(4) in (14). □

Theorem 2.4. *If X and Y are jointly distributed according to (9), then the distribution of $Z=\frac{X}{X+Y}$ is*

$$\begin{aligned} f_Z\left(z;\alpha,\theta;\rho\right) = \alpha^2\Bigg[&(1+\rho)B\left(2,2\alpha\right)\ {}_2F_1\left(2,\alpha+1;2\alpha+2;\frac{2z-1}{z}\right)\\ &+4\rho B\left(2,4\alpha\right)\ {}_2F_1\left(2,2\alpha+1;4\alpha+2;\frac{2z-1}{z}\right)\\ &-2\rho B\left(2,3\alpha\right)\ {}_2F_1\left(2,\alpha+1;3\alpha+2;\frac{2z-1}{z}\right)\\ &-2\rho B\left(2,3\alpha\right)\ {}_2F_1\left(2,2\alpha+1;3\alpha+2;\frac{2z-1}{z}\right)\Bigg] \end{aligned} \tag{17}$$

for $0<z<1$.

Proof. Consider the transformation: $(X,Y)\longrightarrow(R,Z)=\left(X+Y,\frac{X}{X+Y}\right)$ so that

$$\begin{aligned} f_{R,Z}\left(r,z;\alpha,\theta;\rho\right) = \left(\alpha\theta^{\alpha}\right)^2\Bigg\{&\frac{1+\rho}{\left[\left(rz+\theta\right)\left(r-rz+\theta\right)\right]^{2\alpha+1}}+\frac{4\rho\theta^{2\alpha}}{\left[\left(rz+\theta\right)\left(r-rz+\theta\right)\right]^{2\alpha+1}}\\ &-\frac{2\rho\theta^{\alpha}}{\left(rz+\theta\right)^{2\alpha+1}\left(r-rz+\theta\right)^{\alpha+1}}-\frac{2\rho\theta^{\alpha}}{\left(rz+\theta\right)^{\alpha+1}\left(r-rz+\theta\right)^{2\alpha+1}}\Bigg\} \end{aligned}$$

Note that the jacobian of transformation is r, thus

$$f_Z\left(z;\alpha,\theta;\rho\right) = \left(\alpha\theta^{\alpha}\right)^2\left\{\left(1+\rho\right)D\left(1,1\right)+4\rho\theta^{2\alpha}D\left(2,2\right)-2\rho\theta^{\alpha}D\left(2,1\right)-2\rho\theta^{\alpha}D\left(1,2\right)\right\} \tag{18}$$

where

$$D\left(h,k\right) = \int_0^{\infty} r\left(rz+\theta\right)^{-(h\alpha+1)}\left(r-rz+\theta\right)^{-(k\alpha+1)}dr \tag{19}$$

for $h,k\in\{1,2\}$.
Let $u=(1-z)r$. Then $dr=\frac{1}{1-z}du$. One can obtain $D\left(h,k\right)$ as follows

$$D\left(h,k\right) = \int_0^{\infty}\frac{u}{1-z}\left[\frac{uz}{1-z}+\theta\right]^{-(h\alpha+1)}\left[u+\theta\right]^{-(k\alpha+1)}\frac{1}{1-z}du. \tag{20}$$

Using Lemma (2), we have

$$D\left(h,k\right)=\theta^{-(k+h)\alpha}B(2,(h+k)\alpha)_{2}F_{1}\left(2,k\alpha+1;(h+k)\alpha+2;\frac{2z-1}{z}\right) \tag{21}$$

Combining (21) and (18) the result in (17) follows. □

The following figure illustrates the pdf in (17) for specific values: $\rho=0.5,\alpha=2,4$, and 6.

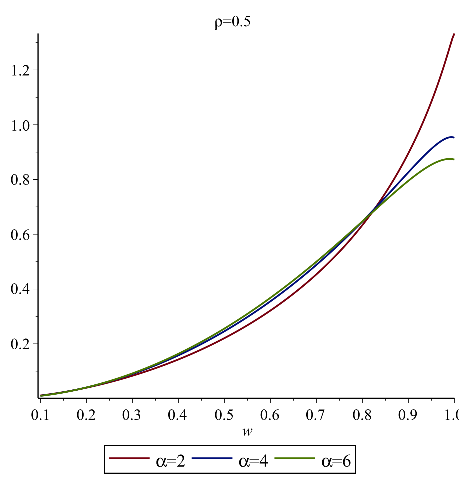

Fig. 3: Graph of the pdf in (17)

Theorem 2.5. *If X and Y are jointly distributed according to (9) then the density function* *of* $R = X + Y$ *is given by*

$$\begin{aligned} f_R(r;\alpha,\theta;\rho) = r(\alpha\theta^{\alpha})^2 \Bigg\{ & (1+\rho)\,\theta^{-(\alpha+1)}(r+\theta)^{-(\alpha+1)} \sum_{j=1}^{\infty}\Bigg[\binom{\alpha+j-1}{j-1}\left(-\frac{r}{\theta}\right)^{j-1} j^{-1} \\ & {}_2F_1\left(j,\alpha+1;j+1;\frac{r}{r+\theta}\right)\Bigg] \\ & + 4\rho\theta^{2\alpha}\theta^{-(2\alpha+1)}(r+\theta)^{-(2\alpha+1)} \sum_{j=1}^{\infty}\Bigg[\binom{2\alpha+j-1}{j-1}\left(-\frac{r}{\theta}\right)^{j-1} j^{-1} \\ & {}_2F_1\left(j,2\alpha+1;j+1;\frac{r}{r+\theta}\right)\Bigg] \\ & - 2\rho\theta^{\alpha}\theta^{-(\alpha+1)}(r+\theta)^{-(\alpha+1)} \sum_{j=1}^{\infty}\Bigg[\binom{2\alpha+j-1}{j-1}\left(-\frac{r}{\theta}\right)^{j-1} j^{-1} \\ & {}_2F_1\left(j,\alpha+1;j+1;\frac{r}{r+\theta}\right)\Bigg] \\ & - 2\rho\theta^{\alpha}\theta^{-(2\alpha+1)}(r+\theta)^{-(2\alpha+1)} \sum_{j=1}^{\infty}\Bigg[\binom{\alpha+j-1}{j-1}\left(-\frac{r}{\theta}\right)^{j-1} j^{-1} \\ & {}_2F_1\left(j,2\alpha+1;j+1;\frac{r}{r+\theta}\right)\Bigg]\Bigg\} \end{aligned}$$

for $0 < r < \infty$.

Proof. Consider the transformation: $(X,Y) \longrightarrow (R,Z) = \left(X+Y, \frac{X}{X+Y}\right)$ so that

$$\begin{aligned} f_{R,Z}(r,z;\alpha,\theta;\rho) = (\alpha\theta^{\alpha})^2 \Bigg\{ & \frac{1+\rho}{[(rz+\theta)(r-rz+\theta)]^{2\alpha+1}} + \frac{4\rho\theta^{2\alpha}}{[(rz+\theta)(r-rz+\theta)]^{2\alpha+1}} \\ & - \frac{2\rho\theta^{\alpha}}{(rz+\theta)^{2\alpha+1}(r-rz+\theta)^{\alpha+1}} - \frac{2\rho\theta^{\alpha}}{(rz+\theta)^{\alpha+1}(r-rz+\theta)^{2\alpha+1}}\Bigg\} \end{aligned}$$

The jacobian of transformation is r, thus

$$f_R(r;\alpha,\theta;\rho) = r(\alpha\theta^{\alpha})^2\left\{(1+\rho)\,G(1,1) + 4\rho\theta^{2\alpha}G(2,2) - 2\rho\theta^{\alpha}G(2,1) - 2\rho\theta^{\alpha}G(1,2)\right\} \quad (22)$$

where

$$G(h,k) = \int_0^1 (rz+\theta)^{-(h\alpha+1)}(r-rz+\theta)^{-(k\alpha+1)}\,dz \quad (23)$$

for $h, k \in \{1,2\}$.
Using Lemma (3), one can obtain $G(h,k)$ as follows

$$
\begin{aligned}
G(h,k) &= \theta^{-(h\alpha+1)} \sum_{j=0}^{\infty} \left[\binom{h\alpha+j}{j} \left(-\frac{r}{\theta}\right)^{j} \int_0^1 z^j \left(-rz+r+\theta\right)^{-(k\alpha+1)} dz \right] \\
&= \theta^{-(h\alpha+1)} \sum_{j=1}^{\infty} \left[\binom{h\alpha+j-1}{j-1} \left(-\frac{r}{\theta}\right)^{j-1} \int_0^1 (z-0)^{j-1} (1-z)^{1-1} \right. \\
&\qquad \left. \left(-rz+r+\theta\right)^{-(k\alpha+1)} dz \right] \\
&= \theta^{-(h\alpha+1)} \sum_{j=1}^{\infty} \left[\binom{h\alpha+j-1}{j-1} \left(-\frac{r}{\theta}\right)^{j-1} (r+\theta)^{-(k\alpha+1)} B(j,1) \right. \\
&\qquad \left. {}_2F_1\left(j, k\alpha+1; j+1; \frac{r}{r+\theta}\right) \right] \\
&= \theta^{-(h\alpha+1)} (r+\theta)^{-(k\alpha+1)} \sum_{j=1}^{\infty} \left[\binom{h\alpha+j-1}{j-1} \left(-\frac{r}{\theta}\right)^{j-1} j^{-1} \right. \\
&\qquad \left. {}_2F_1\left(j, k\alpha+1; j+1; \frac{r}{r+\theta}\right) \right]
\end{aligned} \tag{24}
$$

Combining (22) and (24), the result follows immediately. □

3 Moments

Theorem 3.1. *Let X and Y be jointly distributed according to (9). Then the (a,b)-th product moment of bivariate Lomax density function denoted by ${\mu'}_{a,b;\rho}(X,Y)$ is given by*

$$
\begin{aligned}
{\mu'}_{a,b;\rho}(X,Y) = \Gamma(a+1)\Gamma(b+1)\theta^{a+b} \Big[&\frac{\Gamma(\alpha-a)\Gamma(\alpha-b)}{\Gamma^2(\alpha)} \\
&+ \rho \Big(\frac{\Gamma(2\alpha-a)}{\Gamma(2\alpha)} - \frac{\Gamma(\alpha-a)}{\Gamma(\alpha)} \Big) \Big(\frac{\Gamma(2\alpha-b)}{\Gamma(2\alpha)} - \frac{\Gamma(\alpha-b)}{\Gamma(\alpha)} \Big) \Big]
\end{aligned} \tag{25}
$$

where x, y, α, θ, are all positive, $|\rho| \le 1$ and $\max\{a,b\} < \alpha$.

Proof. By definition, one can expressed the (a, b)-th moment of $f_{X,Y}(x, y; \alpha, \theta; \rho)$ as

$$\begin{aligned}\mu'_{a,b;\rho}(X,Y) = &\int_0^\infty \int_0^\infty x^a y^b \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}} \frac{\alpha\theta^\alpha}{(y+\theta)^{\alpha+1}} dx dy \\ &+ \rho\Bigg[\left(\int_0^\infty \left(2\left(\frac{\theta}{y+\theta}\right)^\alpha - 1\right) \frac{\alpha\theta^\alpha}{(y+\theta)^{\alpha+1}} y^b dy\right) \\ &\left(\int_0^\infty \left(2\left(\frac{\theta}{x+\theta}\right)^\alpha - 1\right) \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}} x^a dx\right)\Bigg].\end{aligned}$$

By Lemma 1, one can show the following integrals:

(1)

$$\int_0^\infty x^a \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}} dx = \alpha\theta^a B(a+1, \alpha-a);$$

(2)

$$\int_0^\infty y^b \frac{\alpha\theta^\alpha}{(y+\theta)^{\alpha+1}} dy = \alpha\theta^b B(b+1, \alpha-b);$$

(3)

$$\int_0^\infty x^a \frac{\alpha\theta^{2\alpha}}{(x+\theta)^{2\alpha+1}} dx = \alpha\theta^a B(a+1, 2\alpha-a);$$

(3) Finally,

$$\int_0^\infty y^b \frac{\alpha\theta^{2\alpha}}{(y+\theta)^{2\alpha+1}} dy = \alpha\theta^b B(b+1, 2\alpha-b);$$

Then the result follows directly. □

Theorem 3.2. *If X and Y are jointly distributed according to 9, then the a-th raw moment of the random variable V is*

$$\mu'_{a;\rho}(V) = \theta^{2a}\Gamma^2(a+1)\left[\frac{\Gamma^2(\alpha-a)}{\Gamma^2(\alpha)} + \rho\left(\frac{\Gamma(2\alpha-a)}{\Gamma(2\alpha)} - \frac{\Gamma(\alpha-a)}{\Gamma(\alpha)}\right)^2\right]. \tag{26}$$

Proof. Notice that

$$E(V^a) = E((X\cdot Y)^a) = E(X^a \cdot Y^a).$$

Putting $b = a$ in (25), the result follows. □

We state the next result without proof since the proof is similar to that of Theorem 3.2.

Theorem 3.3. *If X and Y are jointly distributed according to (9), then a-th raw moment of $W = \frac{X}{Y}$ is*

$$\mu'_{a;\rho}(W) = \Gamma(a+1)\Gamma(1-a)\left[\frac{\Gamma(\alpha-a)\Gamma(\alpha+a)}{\Gamma^2(\alpha)} + \rho\left(\frac{\Gamma(2\alpha-a)}{\Gamma(2\alpha)} - \frac{\Gamma(\alpha-a)}{\Gamma(\alpha)}\right)\left(\frac{\Gamma(2\alpha+a)}{\Gamma(2\alpha)} - \frac{\Gamma(\alpha+a)}{\Gamma(\alpha)}\right)\right] \tag{27}$$

Theorem 3.4. *If X and Y are jointly distributed according to (9), then the a-th raw moment of $Z = \frac{X}{X+Y}$ is*

$$\mu'_{a,\rho}(Z) = \sum_{k=0}^{\infty}\binom{a-1+k}{k}(-1)^k\Gamma(a+k+1)\Gamma(1-a-k)\cdot \left[\frac{\Gamma(\alpha-a-k)\Gamma(\alpha+a+k)}{\Gamma^2(\alpha)} + \rho\left(\frac{\Gamma(2\alpha-a-k)}{\Gamma(2\alpha)} - \frac{\Gamma(\alpha-a-k)}{\Gamma(\alpha)}\right)\left(\frac{\Gamma(2\alpha+a+k)}{\Gamma(2\alpha)} - \frac{\Gamma(\alpha+a+k)}{\Gamma(\alpha)}\right)\right] \tag{28}$$

Proof. Notice that

$$\begin{aligned} E(Z^a) &= E\left(X^a\cdot(X+Y)^{-a}\right) = E\left(\left(\frac{X}{Y}\right)^a\left(\frac{X}{Y}+1\right)^{-a}\right) \\ &= E\left(\left(\frac{X}{Y}\right)^a\sum_{k=0}^{\infty}\binom{a-1+k}{k}(-1)^k\left(\frac{X}{Y}\right)^k\right) \\ &= E\left(\sum_{k=0}^{\infty}\binom{a-1+k}{k}(-1)^k\left(\frac{X}{Y}\right)^{a+k}\right) \\ &= \sum_{k=0}^{\infty}\binom{a-1+k}{k}(-1)^k E\left(W^{a+k}\right) \end{aligned}$$

Using Theorem 3.2, the result in (28) follows. □

Theorem 3.5. *If X and Y are jointly distributed according to (9), then the a-th raw* *moment of $R = X + Y$ is*

$$\mu'_{a;\rho}(R) = \theta^a \sum_{i=0}^{a} \binom{a}{i} \Bigg\{ \Gamma(i+1)\Gamma(a-i+1) \left[\frac{\Gamma(\alpha-i)\Gamma(\alpha-(a-i))}{\Gamma^2(\alpha)} + \rho \left(\frac{\Gamma(2\alpha-i)}{\Gamma(2\alpha)} - \frac{\Gamma(\alpha-i)}{\Gamma(\alpha)} \right) \left(\frac{\Gamma(2\alpha-(a-i))}{\Gamma(2\alpha)} - \frac{\Gamma(\alpha-(a-i))}{\Gamma(\alpha)} \right) \right] \Bigg\} \tag{29}$$

Proof. Since $R^a = (X+Y)^a = \sum_{i=0}^{a} \binom{a}{i} X^i \cdot Y^{a-i}$, then

$$\mu'_{a;\rho}(R) = E(R^a) = \sum_{i=0}^{a} \binom{a}{i} E\left(X^i Y^{a-i}\right) = \sum_{i=0}^{a} \binom{a}{i} \mu'_{i,a-i;\rho}\left(X^i Y^{a-i}\right).$$

By putting $a = i$ and $b = a - i$ in (25), the result follows. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

Balakrishnan, N., and Lai, C. D. (2009). *Continuous bivariate distributions.* New York: Springer.

Gupta, A. K., and Nadarajah, S. (2006). Sums, products and ratios for Mckays bivariate gamma distribution. *Mathematical and computer modelling*, *43*(1), 185-193.

Joarder A.H. et al.(2012). A Bivariate Probability Model Based on Type II Gumbel Random Variable and its Properties. *Technical Report Series 428.* Department of Mathematical Sciences. King Fahd University of Petroleum & Minerals, Saudi Arabia.

Mikosch, T. (2006). Copulas: Tales and facts. *Extremes*, *9*(1), 3-20.

Morgenstern, D. (1956). Einfache Beispiele Zweidimensionaler Verteilungen. *Mitteilingsblatt fur Mathematische Statistik. 8*, 234-235

Mukherjee, S., Jafari, F., & Kim, J. M. (2012). Characterization of Differentiable Copulas. *arXiv preprint arXiv:1210.2953.*

Nadarajah, S. (2005). Sums, products, and ratios for the bivariate gumbel distribution. *Mathematical and computer modelling, 42*(5), 499-518.

Nadarajah, S., & Espejo, M. R. (2006). Sums, products, and ratios for the generalized bivariate Pareto distribution. *Kodai Mathematical Journal*, *29*(1), 72-83.

Nadarajah, S., & Kotz, S. (2007). On the convolution of Pareto and gamma distributions. *Computer Networks, 51*(12), 3650-3654.

Nelsen, R. B. (1999). *An introduction to copulas.* Springer.

Nelsen, R. B. (2003, September). Properties and applications of copulas: A brief survey. *In Proceedings of the First Brazilian Conference on Statistical Modeling in Insurance and Finance,(Dhaene, J., Kolev, N., Morettin, PA (Eds.)), University Press USP: Sao Paulo* (pp. 10-28).

Prieger, J. E. (2002). A flexible parametric selection model for nonnormal data with application to health care usage. *Journal of Applied Econometrics*, 17(4), 367-392.

Rohatgi, V.K. (1976). *An Introduction to Probability Theory Mathematical Statistics.* Wiley, New York.

Sklar, M. (1959). *Fonctions de ré partition à n dimensions et leurs marges.* Université Paris 8.

Schmidt, T. (2007). Coping with copulas. *Copulas-From Theory to Application in Finance*, 3-34.

Trivedi, P. K., and Zimmer, D. M. (2007). Copula modeling: an introduction for practitioners. Now Publishers Inc.